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Random walks in the presence of oriented random forces

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Abstract. The inclusion of the anisotropic vertex is shown to influence the character of random walks. The isotropic fixed point will be unstable. The correction to the diffusion law at large timescales is calculated in the framework of the ε-expansion method.

Recently great interest has arisen in the problem of the classical diffusion of a particle in the random environment. This process is described by the equation

$$[\partial_t - \nabla_i(D\nabla_i - F_i(\boldsymbol{r}))]c(\boldsymbol{r}, t) = 0 \tag{1}$$

where $F_i(\boldsymbol{r})$ is the random vector field with the properties $\langle F_i(\boldsymbol{r})\rangle = 0$ and only simple correlator $\langle F_i(\boldsymbol{r})F_j(\boldsymbol{r}')\rangle \neq 0$. The diffusion law at large timescales has to be determined. In the one-dimensional case this problem was resolved by Sinai (1982). Marinari *et al* (1983) discussed the possible connection of the random walks with the $1/f$ noise problem at $d = 1$ and 2 (their point of view was criticised by Obukov (1983)). At $1 < d \leqslant 2$ with the correlator

$$\langle F_i(\boldsymbol{r})F_j(\boldsymbol{r}')\rangle \sim \delta_{ij}\delta(\boldsymbol{r} - \boldsymbol{r}') \tag{2}$$

the random walk was investigated by Obukhov (1983) and Fisher (1984). The corresponding diagram technique rule has been proposed earlier by Obukhov and Peliti (1983). Fisher *et al* (1985) and Kravtsov *et al* (1985, 1986) have shown that in cases of constrained forces the following Fourier transform of the correlators

$$F_{ij}(\boldsymbol{q}) \equiv \langle F_i(\boldsymbol{q})F_j(0)\rangle \sim \delta_{ij} - \frac{q_i q_j}{q^2} \qquad F_{ij}(\boldsymbol{q}) \sim \frac{q_i q_j}{q^2} \tag{3}$$

should be included.

The interest in all these models is related to the fact that $d = 2$ is the upper critical dimensionality, below which the character of diffusion differs from the classical one. The corresponding corrections can be calculated by means of the ε-expansion method.

Aronovitz and Nelson (1984) investigated the model with $\langle F_i(\boldsymbol{r})\rangle \neq 0$. Also, they pointed out that the model with the fluctuating coefficient D in (1) is not relevant because the problem reduces to the calculation of small corrections within ordinary perturbation theory.

The aim of the present paper is to point out that other correlators must be included because of the possible anisotropy of the medium. We examine here the simplest case of the anisotropy determined by the unit vector $\boldsymbol{n}$.

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Then instead of (2) we have

$$\langle F_i(\boldsymbol{r})F_j(\boldsymbol{r}')\rangle = (g_1(\delta_{ij} - n_in_j) + g_2n_in_j)\delta(\boldsymbol{r} - \boldsymbol{r}') \tag{4}$$

where g_1 and g_2 are the bare coupling constants. We will use the functional integral method by Martin *et al* (1973) (see also De Dominicis and Peliti 1978). Let us write down the functional integral

$$Z = \int \mathscr{D}c(\boldsymbol{r}, t)\delta(\partial_t c - D\nabla^2 c + \nabla_i F_i c) \exp\left(\int \mathrm{d}\boldsymbol{r}\, \mathrm{d}t\, c(\boldsymbol{r}, t)h(\boldsymbol{r}, t)\right) \tag{5}$$

where $c(\boldsymbol{r}, t)$ is the local concentration and $h(\boldsymbol{r}, t)$ is a source field. Eliminating the delta function with the aid of an auxiliary $\bar{c}(\boldsymbol{r}, t)$ field

$$Z = \int \mathscr{D}\bar{c}\mathscr{D}c\, \mathrm{e}^{\mathscr{F}} \qquad \mathscr{F} = \int \mathrm{d}\boldsymbol{r}\, \mathrm{d}t[\mathrm{i}\bar{c}(\partial_t - D\nabla^2 + \nabla_i F_i)c + \mathrm{HC}] \tag{6}$$

and averaging the exponent with the correlators (4), we obtain

$$\mathscr{F} = \int \mathrm{d}\boldsymbol{r}\, \mathrm{d}t[\mathrm{i}\bar{c}(\partial_t - D\nabla^2)c - \tfrac{1}{2}(g_1(\delta_{ij} - n_in_j) + g_2n_in_j)$$
$$\times \int \mathrm{d}t'(\nabla_i\bar{c}(\boldsymbol{r}, t))c(\boldsymbol{r}, t)(\nabla_j\bar{c}(\boldsymbol{r}, t'))c(\boldsymbol{r}, t') + \mathrm{HC}]. \tag{7}$$

After the Fourier transformation

$$\mathscr{F} = \int \frac{\mathrm{d}\omega}{2\pi}\left\{\int \frac{\mathrm{d}^d q}{(2\pi)^d}[\mathrm{i}\bar{C}_{q\omega}(-\mathrm{i}\omega + Dq^2)C_{q\omega} + h_{-q,-\omega}C_{q\omega}] + \tfrac{1}{2}(g_1(\delta_{ij} - n_in_j) + g_2n_in_j)\right.$$
$$\left.\times \int \frac{\mathrm{d}^d q_1 \ldots \mathrm{d}^d q_4}{(2\pi)^{4d}} q_{1i}q_{3j}\bar{C}_{q_1\omega}C_{q_2\omega}\bar{C}_{q_3\omega}C_{q_4\omega}\delta(\boldsymbol{q}_1 + \boldsymbol{q}_3 - \boldsymbol{q}_2 - \boldsymbol{q}_4)\right\}. \tag{8}$$

In the one-loop approximation the corrections to the vertices are given by the diagrams in figures 1(*a*) and (*b*). Performing the calculation of diagrams, we get the renormalisation group equations

$$\frac{\mathrm{d}\tilde{g}_1}{\mathrm{d}\xi} = \varepsilon\tilde{g}_1 - \tilde{g}_1\tilde{g}_2 \qquad \frac{\mathrm{d}\tilde{g}_2}{\mathrm{d}\xi} = \varepsilon\tilde{g}_2 - \tilde{g}_1\tilde{g}_2 \tag{9}$$

$$\xi = \frac{1}{\varepsilon}\left[\left(\frac{q_{\max}}{\max\{q, (\omega/D)^{1/2}\}}\right)^{\varepsilon} - 1\right] \tag{10}$$

where $\varepsilon = 2 - d$, $\tilde{g}_\alpha = q_{\max}^{-\varepsilon}g_\alpha/4\pi D^2$ is a dimensionless coupling constant, $q_{\max}$ is the upper cut-off of the integrals. Equations (9) and (10) have fixed point at $\tilde{g}_1 = \tilde{g}_2 = \varepsilon$. The trajectories on the phase plane $\tilde{g}_1$, $\tilde{g}_2$ at $\varepsilon > 0$ are presented in figure 2. The

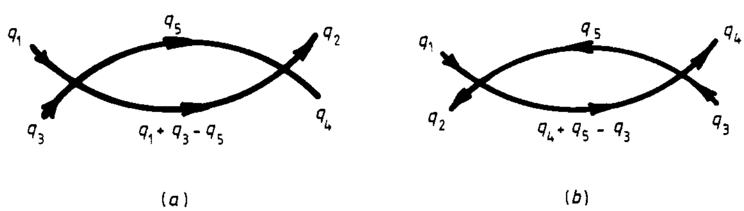

Figure 1. The vertex corrections.

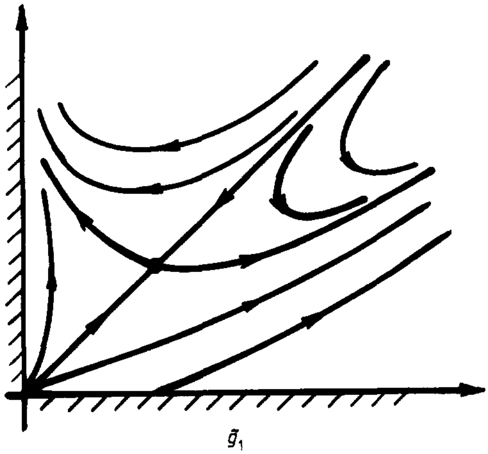

Figure 2. The phase trajectories.

hatching shows the stability borders of the Hamiltonian. As we can see from figure 2, the fixed point $g_1 = g_2 = \varepsilon$ is unstable (saddle point). If there is an inequality for the bare coupling $\tilde{g}_2^0 > \tilde{g}_1^0$, we have asymptotically at $\xi \to \infty$

$$\tilde{g}_1 = \tilde{g}_1^0 \exp(-2\tilde{g}_2\xi) \to 0 \qquad \tilde{g}_2 = \tilde{g}_2^0 (q_{\max}/q)^{\varepsilon}. \tag{11}$$

Here $q = \max\{q_0, (\omega/D)^{1/2}\}$. If $\tilde{g}_2^0 < \tilde{g}_1^0$, we have asymptotically

$$\tilde{g}_1 = \tilde{g}_1^0 (q_{\max}/q)^{\varepsilon} \qquad \tilde{g}_2 = \tilde{g}_2^0 \exp(-2\tilde{g}_1\xi) \to 0. \tag{12}$$

Now we calculate the correction to the correlator; see figure 3. In the vertices we put the renormalised coupling $\tilde{g}_2$ (at $g_2^0 > g_1^0$). The analytical expression has the form

$$\sum(\boldsymbol{q}, \omega) = \int \frac{\mathrm{d}^d q_1\, \mathrm{d}^d q_2}{(2\pi)^{2d}}\, g_2^2(\boldsymbol{q}_1 + \boldsymbol{q}_2 + \boldsymbol{q}) n_i n_j n_k n_l$$
$$\times \frac{q_i (q_{1j} + q_{2j}) q_{2k} (q_{1l} + q_l)}{(-\mathrm{i}\omega + D(\boldsymbol{q}_1 + \boldsymbol{q})^2)(-\mathrm{i}\omega + D q_2^2)(-\mathrm{i}\omega + D(\boldsymbol{q}_1 + \boldsymbol{q}_2)^2)}. \tag{13}$$

With logarithmic accuracy it follows from (13) that

$$\sum(\boldsymbol{q}, \omega) = 4D\tilde{g}_2^2 n_i n_j q_i q_j \xi. \tag{14}$$

The correlator $\langle r_\alpha(t) r_\beta(t)\rangle$ can be calculated with the aid of the Green function

$$\langle r_\alpha(t) r_\beta(t)\rangle = -\frac{\mathrm{d}^2}{\mathrm{d}q_\alpha\, \mathrm{d}q_\beta} \int \frac{\mathrm{d}\omega}{2\pi} \mathrm{e}^{-\mathrm{i}\omega t} G(\boldsymbol{q}, \omega) \bigg|_{q=0} \tag{15}$$

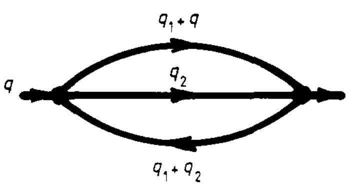

Figure 3. The self-energy diagram.

where

$$G^{-1}(\boldsymbol{q}, \omega) = -\mathrm{i}\omega + Dq^2 - \textstyle\sum (\boldsymbol{q}, \omega). \tag{16}$$

After the substitution of (14) and (16) in (15) we obtain

$$\langle r_\alpha(t) r_\beta(t) \rangle = Dt(\delta_{\alpha\beta} - \tilde{g}_2^{02} n_\alpha n_\beta t^\varepsilon). \tag{17}$$

It follows from (17) that in the fixed direction the character of diffusion changes drastically; the corresponding correction $\sim t^{1+\varepsilon}$.

In summary, the inclusion of the anisotropic vertex seriously influences the character of diffusion. The anisotropy is more strongly manifest at large timescales. This leads to the correction of the diffusion law.

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